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Pseudo Difference Posets and Pseudo Boolean D-Posets

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The definitions of pseudo difference posets, pseudo boolean D-posets, and D-ideals are introduced. It is proved that pseudo difference posets are algebraically equivalent to pseudo effect algebras and pseudo boolean D-posets are algebraically equivalent to pseudo MV-algebras. In pseudo difference lattices, a D-ideal is equal to a Riesz ideal. At the same time, some good properties are obtained.

KEY WORDS: pseudo difference posets; pseudo boolean D-posets; MV-algebras; D-ideals.

1. INTRODUCTION AND BASIC DEFINITIONS

With the development of the theory of quantum logics, new algebraic structures have been proposed as their models. As a quantum structure generalizing orthomodular lattices, orthomodular posets, and orthoalgebras, effect algebras in which the primary operation is partial sum, are regarded as a mathematical model of quantum logic (Foulis and Bennett, 1994; Foulis *et al.*, 1992; Kalmbach, 1983). From a completely different starting point, Kôpka and Chovanec(1994) defined D-posets as an axiomatic model for quantum logics, where the primary operation is partial difference. This is important for modelling unsharp measurement in quantum mechanics (Dvurečenskij and Pulmannová, 1994). Moreover, the two models are equivalent.

In the study of quantum logics, MV-algebras play an analogous role to that of Boolean algebras in classics logic (Chang, 1958; Dvurečenskij and Pulmannová, 2000). By the partial sum operations, we see that effect algebras have close relation to MV-algebras in which the primary operation is total sum. Along the partial difference direction, there is a special kind of D-posets, namely Boolean D-posets, which are algebraically equivalent to MV-algebras (Chovanec and Kôpka, 1997).

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From the interplay between partial sum operation and partial difference operation, many good results are obtained in the quantum logics. Particularly, by the structure of D-posets, many problems about noncommutative probability theory and quantum measurements can be solved in some sense (Dvurečenskij and Pulmannová, 2000).

Recently, several new kinds of quantum models appeared. Among these models, pseudo-MV-algebras were proposed by dropping community in the total sum operation of MV-algebras (Georgescu and lorgulescu, 2001). Similarly, pseudo effect algebras were derived from effect algebras through getting rid of community in the partial sum operation (Dvurečenskij and Vetterlein, 2001a,b). The relationship between above two models is similar to that between MV-algebras and effect algebras. Since there is no community, it is difficult to define the corresponding difference posets. From the positive cone of po-group point of view, Dvurečenskij *et al.* have discussed unbounded situations (namely, generalized pseudo effect algebras), and got some good results (Dvurečenskij and Vetterlein 2000a,b).

In this note, we mainly discuss bounded conditions. We introduce pseudo difference posets, especially pseudo boolean D-posets, and prove that a pseudo difference poset is algebraically equivalent to a pseudo effect algebra. In particular, we directly show that a pseudo boolean D-poset is algebraically equivalent to a pseudo MV-algebra. At the same time, we give some good properties of them in detail. In the end, we present the concept of a D-ideal (Avallone and Vitolo, 2003) in pseudo difference lattices and obtain that it is equivalent to a Riesz ideal in lattice order pseudo effect algebras. Hence it is invariant under generalized Sasaki projection (Pulmannová, 2003).

Definition 1.1. (Dvurečenskij and Vetterlein, 2001a). A structure $(E; +, 0, 1)$, where $+$ is a partial order binary operation and 0 and 1 are constants, is called a pseudo effect algebra if for all $a, b, c \in E$, the following hold.

(PE1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exists, and in this case, $(a + b) + c = a + (b + c)$.
(PE2) There is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.
(PE3) If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.
(PE4) If $1 + a$ or $a + 1$ exists, then $a = 0$.

Remark 1.2. Let $(E; +, 0, 1)$ be a pseudo effect algebra.

(i) $a \leq b$ iff $a + c = b$ for some $c \in E$.
(ii) For all $a, b, c \in P$, $a + b = a + c$ implies $b = c$, and $b + a = c + a$ implies $b = c$ (cancellation laws).
(iii) If $a + b$ exists, $a_1 \leq a$, and $b_1 \leq b$, then $a_1 + b_1$ exists.

(iv) Assume $b + c$ exists. Then $a \leq b$ iff $a + c$ exists and $a + c \leq b + c$. Assume $c + b$ exists. Then $a \leq b$ iff $c + a$ exists and $c + a \leq c + b$.

Definition 1.3. (Kôpka and Chovanec, 1994). Let $(P, \leq)$ be a poset with the least element 0 and the largest element 1. Let $-$ be a partial binary operation on P, such that $b - a$ is defined iff $a \leq b$. Then $(P; \leq, -, 0,1)$ is called a difference poset (D-poset) if the following conditions are satisfied:

(Dl) For any $a \in P$, $a - 0 = a$.
(D2) If $a \leq b \leq c$, then $c - b \leq c - a$ and $(c - a) - (c - b) = b - a$.

Definition 1.4. (Chovanec and Kôpka, 1997). A poset P with the least element 0 and the largest element 1 is called a Boolean D-poset, if there is a binary operation $\ominus$ on P, satisfying the following conditions:

(BD1) $a \ominus 0 = a$ for any $a \in P$.
(BD2) $a \ominus (a \ominus b) = b \ominus (b \ominus a)$ for any $a, b \in P$.
(BD3) $a, b \in P, a \leq b$ implies that $c \ominus b \leq c \ominus a$ for any $c \in P$.
(BD4) $(a \ominus b) \ominus c = (a \ominus c) \ominus b$ for any $a, b, c \in P$.

Definition 1.5. (Georgescu and lorgulescu, 2001). A structure $(M; \oplus, -, \sim, 0,1)$, where $\oplus$ is a binary, $-$ and $\sim$ are unary operations, and 0,1 are constants, is called a pseudo-MV algebra, if the following axioms hold in it.

(Al) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.
(A2) $x \oplus 0 = 0 \oplus x = x$.
(A3) $x \oplus 1 = 1 \oplus x = 1$.
(A4) $\tilde{1} = 0; \bar{1} = 0$.
(A5) $(\bar{x} \oplus \bar{y})^{\sim} = (\tilde{x} \oplus \tilde{y})^{-}$.
(A6) $x \oplus (\tilde{x} \odot y) = y \oplus (\tilde{y} \odot x) = (x \odot \bar{y}) \oplus y = (y \odot \bar{x}) \oplus x$.
(A7) $x \odot (\bar{x} \oplus y) = (x \oplus \tilde{y}) \odot y$.
(A8) $x^{-\sim} = x$.

Here, for any $a, b \in M$, we put $a \odot b = (\bar{b} \oplus \bar{a})^{\sim}$.

Lemma 1.6. *(Georgescu and lorgulescu, 2001). Let M be a pseudo MV-algebra.*

Then

(i) $y \odot x = (\widetilde{x} \oplus \widetilde{y})^{-}$.
(ii) $x^{\sim -} = x$.
(iii) $(x \oplus y)^{-} = \bar{y} \odot \bar{x}; (x \oplus y)^{\sim} = \tilde{y} \odot \tilde{x}$.
(iv) $(x \odot y)^{-} = \bar{y} \oplus \bar{x}; (x \odot y)^{\sim} = \tilde{y} \oplus \tilde{x}$.
(v) $x \oplus y = (\bar{y} \odot \bar{x})^{\sim} = (\tilde{y} \odot \tilde{x})^{-}$.

(vi) $\tilde{x} \odot y \oplus \tilde{y} = \tilde{y} \odot x \oplus \tilde{x}$.
(vii) $x \odot (\bar{x} \oplus y) = y \odot (\bar{y} \oplus x)$.
(viii) $x \odot (y \odot z) = (x \odot y) \odot z$.

Definition 1.7. (Pulmannová, 2003). Let P be a pseudo effect algebra. A nonempty subset I of P is said to be an ideal if

(I1) $x \in I$, and $y \leq x$, implies $y \in I$.
(I2) If $x, y \in I$ and $x + y$ is defined in P, then $x + y \in I$.

An ideal I is said to be a Riesz ideal if
(R0) $x \in I, a, b \in P, x \leq a + b$ implies that there exists $c, d \in I, x \leq c + d, c \leq a, d \leq b$.

2. PSEUDO DIFFERENCE POSETS

Definition 2.1. Let $(P, \leq)$ be a poset with the least element 0 and the largest element 1. Let /, \ be two partial binary operations such that for $a, b \in P$, b/a is defined iff $b\backslash a$ is defined iff $a \leq b$. Then $(P; \leq, /, \backslash, 0, 1)$ is called a pseudo difference poset (PD-poset) if the following conditions are satisfied:

(PD1) For any $a \in P$, $a/0 = a\backslash 0 = a$.
(PD2) If $a \leq b \leq c$, then $c/b \leq c/a$ and $c\backslash b \leq c\backslash a$, and we have $(c/a)\backslash(c/b) = b/a$, and $(c\backslash a)/(c\backslash b) = b\backslash a$.

Remark 2.2. If $(P; \leq)$ is a lattice, then P is called a PD-lattice.

Example 2.3. Let P be a partial ordered group and u a positive element of G (Fuchs, 1963). $\Gamma(P, u) = [0, u]$, it is easy to check that $\Gamma(P, u)$ is a PD-poset. If P is a l-group, then $\Gamma(P, u)$ is a PD-lattice. Where if $a \leq b$, then $b/a = b - a$ and $b\backslash a = -a + b$, for any $a, b \in [0, u]$.

Proposition 2.4. *Let $(P; \leq, /, \backslash, 0, 1)$ be a PD-poset. The following properties hold:*

(i) *If $a \leq b$, then $b/a, b\backslash \leq b$ and $b\backslash(b/a) = a, b/(b\backslash a) = a$.*
(ii) *For any $a \in P$, $a/a = a\backslash a = 0$.*
(iii) *If $a, b \leq c$, and $c/a = c/b$ or $c\backslash a = c\backslash b$, then we have $a = b$*
(iv) *If $a \leq b \leq c$, then $b/a \leq c/a$ and $b\backslash a \leq c\backslash a$, and we have $(c/a)/(b/a) = c/b$ and $(c\backslash a)\backslash(b\backslash a) = c\backslash b$.*
(v) *If $c \leq a, b$, and $a/c = b/c$ or $a\backslash c = b\backslash c$, then $a = b$.*
(vi) *If $b \leq c$ and $a \leq c\backslash b$, then $b \leq c/a$ and $(c/a)\backslash b = (c\backslash b)/a$.*

Proof:

(i) Let $a \leq b$, then $0 \leq a \leq b$, by (PD1) and (PD2), we have $b/a \leq b/0 = b$ and $b\backslash a \leq b\backslash 0 = b$. Further, by (PD2), $b\backslash(b/a) = a/0 = a$, $b/(b\backslash a) = a\backslash 0 = a$.

(ii) Since 0 is the smallest element, then $a\backslash 0 \leq a$ by (i). Hence, $0 = a/(a\backslash 0) \geq a/a$ by PD2. That is $a/a = 0$. Similarly, we have $a/a = 0$.

(iii) Let $a, b \leq c$, and $c/a = c/b$, by (i) then $a = c\backslash(c/a) = c\backslash(c/b) = b$. For another case, we can prove similarly.

(iv) Let $a \leq b \leq c$, by PD2 and (i), then $b/a = (c/a)\backslash(c/b) \leq (c/a)$. Further, since $c/b \leq (c/a)$, then $c/b = (c/a)/[(c/a)\backslash(c/b)] = (c/a)/(b/a)$. Similarly, we can prove another expression.

(v) Since $c \leq a, b \leq 1$, then by (iv), $1/b = (1/c)/(b/c) = (1/c)/(a/c) = I/a$, hence, $a = b$ by (iii). Similarly to another case.

(vi) Since $a \leq c\backslash b \leq c$, by (PD2), then $b \leq c/a$. Hence, $(c\backslash b)/a = (c/a)\backslash(c/(c\backslash b))$, i.e., $(c\backslash b)/a = (c/a)\backslash b$.

□

Definition 2.5. Let $(P; \leq, /, \backslash, 0,1)$ be a PD-poset. If $/ = \backslash$, then we say P is communicative.

Proposition 2.6. *Let $(P; \leq, /, \backslash, 0,1)$ be a PD-poset. If P is communicative, let $- = / = \backslash$, then $(P; \leq, -, 0,1)$ is a D-poset.*

Conversely, let $(P; \leq, -, 0,1)$ be a D-poset. Then $(P; \leq, -, -, 0,1)$ is a commutative PD-poset.

The proof is evident.

The proofs of the following two propositions are similar to that for unbounded situations (Dvurečenskij and Vetterlein, 2000b). For the completeness, we show them in detail.

Proposition 2.7. *Let $(P; \leq, /, \backslash, 0,1)$ be a PD-poset. If $a, b \in P$, and $a \leq b$, define the partial binary + on pairs $(b/a, a)$, such that $b/a + a = b$. Then for any $a, b, c \in P$, $a + b$ is defined and equals c iff $b \leq c$ and $c/b = a$ iff $a \leq c$ and $c\backslash a = b$. (1)*

Furthermore, $(P; +, 0,1)$ is a pseudo effect algebra whose order coincide with the original.

In addition, let $/_r$ and $\backslash_r$ be partial binary operations such that for $a, b, \in P$, $b/_r a$ is defined iff $a \leq b$, in which case, we require $(b/_r a) + a = a + (b\backslash_r a) = b$ to hold. Then $/_r = /$ and $\backslash_r = \backslash$.

Proof: First, + is well defined. Indeed, if for $a_1, a_2, b_1, b_2 \in P$, $b_1/a_1 = b_2/a_2$, $a_1 = a_2$, then by Proposition 2.4(v), we have $b_1 = b_2$.

Now, we prove (1) is true. Let $a, b, c \in P$, by definition, we have $a + b = c$ iff $b \leq c$ and $a = c/b$. Suppose $b \leq c$, $c/b = a$, then $a \leq c$ and $c\backslash a = c\backslash(c/b) = b$ by proposition 2.4(i). If $a \leq c$ and $c\backslash a = b$, again by proposition 2.4(i), we have $b \leq c$ and $c/b = a$. Now we prove (PE1)–(PE4) is true.

(PE1). Suppose $(a + b) + c$ exists and $(a + b) + c = y$. By (1), it follows that $a + b = y/c$ and $b = (y/c)\backslash a$. Since $a \leq y/c \leq y$, then $c = y\backslash(y/c) \leq y\backslash a$; from Proposition 2.4 (iv), we have $a = y/(y\backslash a) = (y/c)/[(y\backslash a)/c]$. Hence, $(y\backslash a)/c = (y/c)\backslash a = b$ by Proposition 2.4 (i). Again by(1), then $y = a + (b + c)$. Similarly, if the latter expression exists, we can show that $(a + b) + c$ also exists, and both terms are equal.

(PE2). Suppose $a + b$ is defined. Then a, $b \leq a + b$ by (1), and we have $((a + b)/a) + a = b + ((a + b)\backslash b) = a + b$.
Uniqueness is obvious by (1).

(PE3). For any $a \in P$, $a \leq 1$, then $1\backslash a \leq 1$, $1/a \leq 1$ and $a = 1/(1\backslash a) = 1\backslash(1/a)$, hence, by (1), $a + 1\backslash a = 1 = 1/a + a$. Uniqueness is similar to the above.

(PE4). Suppose $1 + a$ or $a + 1$ exists, and $1 + a = c$, by (1), then $1 \leq c$. Since 1 is the largest element, then $c \leq 1$. So $c = 1$, and $a = 1\backslash 1 = 0$ by proposition 2.4 (ii).

Now, we prove the orders of P as a PD-poset and as a PE-algebra coincide.

Suppose a $a \leq_{PD} b$, then by (1), $a + b\backslash a = b$, hence $a \leq_{PE} b$. Conversely, if for some $c \in P$, $a + c = b$, then we have a $a \leq_{PD} b$ by (1).

Finally, for $a, b \in P$, b/a is defined iff $a \leq b$. By (1), then $(b/a) + a = b$, So $b/_r a = b/a$. Thus, $/_r = /$, and analogously, $\backslash_r = \backslash$. □

Proposition 2.8. *Let $(P; +, 0,1)$ be a pseudo effect algebra with the order $\leq$, / and \ be partial binary operations such that, for $a, b \in P$, b/a is defined iff $b\backslash a$ is defined iff $a \leq b$, in which case, we require $(b/a) + a = a + b\backslash a = b$ (2) to hold.*

Then $(P; \leq, /, \backslash, 0,1)$ is a PD-poset.

Further, let us define the partial binary operation $+_r$ for all pairs $(b/a, a)$, where $a \leq b$, by setting $(b/a) +_r a = b$. Then $+_r = +$.

Proof: Obviously, $(P; \leq,0,1)$ is a poset with 0 as the smallest element and 1 as the largest element. Let $a, b \in P$, $b\backslash a$ is defined iff $a \leq b$.

Since $0 \leq a$, then $a/0 + 0 = a = 0 + a\backslash 0 = a$ by (2), namely, $a/0 = a$ and $a\backslash 0 = a$. Hence, (PD1) holds. Let $a \leq b \leq c$. Then $b = a_1 + a$ for some $a_1 bP$, and from (2) we get $(c/b) + (a_1 + a) = c/b + b = c/a + a$, so $c/b \leq c/a$ by Remark 1.2(iv). Moreover, we have $c/b + (c/a)\backslash(c/b) + a = c/b + b/a + a$ by (2). So $b/a = (c/a)\backslash(c/b)$ by the cancellation law of P. The second expression is proved similarly. Thus, (PD2) is true. The operation $+_r$, as seen from the cancellation of P, is well defined. Now, for $a, b, c \in E$, $c + a$ is defined and equals b iff $a \leq b$ and $b/a = c$. It follows that $+_r$ coincide with $+$.

Evidently, the relationship between PD-posets and pseudo effect algebras is similar to the relationship between D-posets and effect algebras.

From Proposition 2.7 and Proposition 2.8, we can conclude : □

Proposition 2.9. *Let P be a PD-poset(pseudo effect algebra), then:*

(i) For any $a, b, c \in P$, $a + b \leq c$ iff $a \leq c/b$ iff $b \leq c\backslash a$.
(ii) If $b + a \leq c$, then $(c\backslash b)\backslash a = c\backslash(b + a)$, and $(c/a)/b = c/(b + a)$.

Proposition 2.10. *Let P be a PD-poset. For any $a, b, c \in P$, if $a \leq c$, $b \leq c$ and $a \vee b$ exists, then*

(i) $(c\backslash a) \wedge (c\backslash b)$ exists and $c\backslash(a \vee b) = (c\backslash a) \wedge (c\backslash b)$.
$(c/a) \wedge (c/b)$ exists, and $c/(a \vee b) = (c/a) \wedge (c/b)$.
(ii) $((a \vee b)\backslash a) \wedge ((a \vee b)\backslash b) = 0$.
$((a \vee b)/a) \wedge ((a \vee b)/b) = 0$.

Proof:

(i) Suppose $a \leq c, b \leq c$, then $a \leq a \vee b \leq c, b \leq a \vee b \leq c$, hence, $c\backslash (a \vee b) \leq c\backslash a, c\backslash(a \vee b) \leq c\backslash b$. For any $d \in P$, $d \leq c\backslash a$, $d \leq c\backslash b$, then $d \leq c\backslash a \leq c$, $d \leq c\backslash b \leq c$, hence, $c/(c\backslash a) \leq c/d(c\backslash b) \leq c/d$, that is, $a \leq c/d \leq c, b \leq c/d \leq c$, then $(a \vee b) \leq c/d \leq c$. Thus, $c\backslash(c/d) \leq c$ $(a \vee b)$, i.e., $d \leq c\backslash(a \vee b)$. Hence, $(c\backslash a) \wedge (c\backslash b)$ exists and $c\backslash(a \vee b) = (c\backslash a)/\backslash(c\backslash b)$. Similarly, $(c/a) \wedge (c/b)$ exists and $c/(a \vee b) = (c/a) \wedge (c/b)$.

□

(ii) From (i), we only need to take $c = a \vee b$.

Proposition 2.11. *Let P be a PD-lattice. For any $a, b, c \in P$,*

(i) If $a \leq c, b \leq c$, then $c\backslash(a \wedge b) = (c\backslash a) \vee (c\backslash b)$.

$$c/(a \wedge b) = (c/a) \vee (c/b).$$

$$(a \vee b)\backslash(a \wedge b) = ((a \vee b)\backslash a) \vee ((a \vee b)\backslash b).$$

$$(a \vee b)/(a \wedge b) = ((a \vee b)/a) \vee ((a \vee b)/b).$$

(ii) If $c \leq a, c \leq b$, then $(a \wedge b)\backslash c = (a\backslash c) \wedge (b\backslash c)$.

$$(a \wedge b)/c = (a/c) \wedge (b/c).$$

$$(a\backslash(a \wedge b)) \wedge (b\backslash(a \wedge b)) = 0.$$

$$(a/(a \wedge b)) \wedge (b/(a \wedge b)) = 0.$$

(iii) If $c \leq a, c \leq b$, then $(a \vee b)\backslash c = (a\backslash c) \vee (b\backslash c)$.

$$(a \vee b)/c = (a/c) \vee (b/c).$$

Proof:

(i) Since $a \wedge b \leq a, b \leq c$, then $c\backslash a \leq c\backslash a \wedge b$, $c\backslash b \leq c\backslash a \wedge b$. Let $c\backslash a \leq d$, $c\backslash b \leq d$, then by Proposition 2.7, there exists e, f such that $e + c\backslash a = d$, $f + c\backslash b = d$. Hence $c\backslash a = d\backslash e$, $c\backslash b = d\backslash f$. By $a + (c\backslash a) = c$, $b + (c\backslash b) = c$, we have $a + (d\backslash e) = c$, $b + (d\backslash f) = c$. i.e., $a = c/(d\backslash e)$, $b = c/(d\backslash f)$. Hence, $a \wedge b = c/(d\backslash e) \wedge c/(d\backslash f) = c/(d\backslash e \vee d\backslash f)$ by Proposition 2.10(i). Hence, $d\backslash e \vee d\backslash f = c\backslash(a \wedge b) \leq d$. Thus, $(c\backslash a) \vee (c\backslash b) = c\backslash(a \wedge b)$. Similarly, $(c/a) \vee (c/b) = c/(a \wedge b)$. Let $c = a \vee b$ we can conclude another two expressions.

(ii) Since $c \leq a, b$, then $c \leq a \wedge b \leq a, b$, so $(a \wedge b)\backslash c \leq a\backslash c, b\backslash c$. Let $d \leq a\backslash c$, $d \leq b\backslash c$, then $c + d$ exists and $c + d \leq a$, $c + d \leq b$, hence, $c + d \leq a \wedge b$. It follows that $d \leq a \wedge b\backslash c$. So, $(a \wedge b)\backslash c = (a\backslash c) \wedge (b\backslash c)$. For another two cases, we only have to take $c = a \wedge b$.

(iii) From $c \leq a$, $b \leq a \vee b$, then $a\backslash c \leq a \vee b\backslash c$, $b\backslash c \leq a \vee b\backslash c$. And $(a\backslash c) \vee (b\backslash c) \leq a \vee b\backslash c$. By Proposition 2.10(i), we have $((a \vee b)\backslash c)\backslash((a\backslash c) \vee (b\backslash c)) = (((a \vee b)\backslash c)\backslash(a\backslash c)) \wedge (((a \vee b)\backslash c)\backslash(b\backslash c)) = ((a \vee b)\backslash a) \wedge ((a \vee b)\backslash b) = 0$. So $a \vee b\backslash c = (a\backslash c) \vee (b\backslash c)$, and $(a \vee b)/c = (a/c) \vee (b/c)$.

□

Proposition 2.12. *Let P be a PD-poset. Let a, b_i, $i \in I$, be elements in P, such that $b = \wedge b_i$.*

If $a \leq b_i$, for any $i \in I$, then $b/a = \wedge\{b_i/a|\ i \in I\}$, and $b\backslash a = \wedge\{b_i\backslash a|\ i \in I\}$.

Proof: Since for any $i \in I$, $a \leq b \leq b_i$, then $b/a \leq b_i/a$. Take any $d \in P$, such that $b_i/a \geq d$, by Proposition 2.9(i), then for any $i \in I$, $d + a \leq b_i$, hence $d + a \leq b$, which implies that $d \leq b/a$. So, $b/a = \wedge\{b_i/a|\ i \in I\}$. Similarly, we can conclude another case. □

Proposition 2.13. *Let P be a PD-poset. Let a, b_i, $i \in I$, be elements in P, such that $b = \vee b_i \in P$.*

If for any $i \in I$, $b_i \leq a$, then $a/b = \wedge\{a/b_i|\ i \in I\}$, and $a\backslash b = \wedge\{a\backslash b_i|\ i \in I\}$.

Proof: Since for any $i \in I$, $b_i \leq b \leq a$, then $a/b \leq a/b_i$. Take $d \in P$, such that $d \leq a/b_i$, by Proposition 2.9(i), then $d + b_i \leq a$, i.e., $b_i \leq a\backslash d$, hence $b \leq a\backslash d$. It follows that $d \leq a/b$. Thus, $a/b = \wedge\{a/b_i | i \in I\}$. Analogously, $a\backslash b = \wedge\{a\backslash b_i |\ i \in I\}$. □

Theorem 2.14. *Let P be a PD-lattice. Then there are two total binary difference operations* $\Subset$ *and* $\Supset$ *on P such that the following properties hold.*

(PB1) $a \Subset 0 = a \Supset 0 = a$ *for any* $a \in P$.
(PB2) *For any* $a, b \in P$, $a \Subset (a \Supset b) = b \Subset (b \Supset a)$.
$a \Supset (a \Subset b) = b \Supset (b \Subset a)$.
(PB3) *If* $a, b \in P$, $a \leq b$, *then* $c \Subset b \leq c \Subset a$, $c \Supset b \leq c \Supset a$, *for any* $c \in P$.
(PB4) *If* $a \leq b \leq c$, *then* $(c \Supset a) \Subset (c \Supset b) = b \Supset a$.
$(c \Subset a) \Supset (c \Subset b) = b \Subset a$.

Conversely, let P be a poset with the largest element 1. Let $\Subset$, $\Supset$ *be binary operations on P with the properties (PB1)–(PB4). Then P is a PD-lattice.*

Proof: Let $a \Supset b = a/(a \wedge b)$, $a \Subset b = a\backslash(a \wedge b)$. It is easy to see that the binary operations $\Subset$, $\Supset$ has the following properties.

(i) If $a \leq b$, then $b \Subset a = b\backslash a$, $b \Supset a = b/a$.
(ii) For any $a, b \in P$, $b \Subset a \leq b$, $b \Supset a \leq b$.
(iii) For any $a, b \in P$, $b \Subset (b \Supset a) = a \wedge b = b \Supset (b \Subset a)$.
(iv) If $b \leq a$, then $b \Subset a = 0$, $b \Supset a = 0$.
(v) For any $a, b \in P$, $a \wedge b = 0$ iff $b \Subset a = b$ iff $b \Supset a = b$.
(vi) For any $a, b \in P$, $c \Subset (a \wedge b) = (c \Subset a) \vee (c \Subset b)$, $c \Supset (a \wedge b) = (c \Supset a) \vee (c \Supset b)$.

From these, we can conclude that (PB1), (PB2), (PB3) hold.

For (PB4), by definition, $(c \Supset a) \Subset (c \Supset b) = (c \Supset a) \setminus (c \Supset a) \wedge (c \Supset b)$. Since $a \leq b \leq c$, then it is equal to $(c/a) \setminus (c/a) \wedge (c/b) = (c/a) \setminus c/(a \vee b) = (c/a) \setminus (c/b) = b/a = b \Supset a$ by Proposition 2.10(i) and Definition 2.1.

Conversely, it is easy to see that $(P; \leq, \Subset, \Supset, 0, 1)$ is a PD-poset from (PB1), (PB3) and (PB4). Now, we prove $(P, \leq)$ is a lattice. Evidently, from Proposition 2.4(1), $a \Subset (a \Supset b) \leq a, b$. Let $u \in P$, $u \leq a, b$. First, we note that $u \Supset b = 0$. Indeed, since $0 \Supset (0 \Subset u) = u \Supset (u \Subset 0) = u \Supset u$, hence, $u \Supset u = 0$. Thus, $u \Supset b \leq u \Supset u = 0$, i.e., $u \Supset b = 0$. So we have $u = u \Subset 0 = u \Subset (u \Supset b) = b \Subset (b \Supset u) \leq b \Subset (b \Supset a) = a \Subset (a \Supset b)$. Hence, $a \Subset (a \Supset b)$ is the infimum of a and b. Thus, $b \Subset (b \Supset a) = a \Subset (a \Supset b) = a \wedge b$. Similarly, $a \Supset (a \Subset b) = b \Supset (b \Subset a) = a \wedge b$. Define $\bar{a} = 1 \Supset a$, $\tilde{a} = 1 \Subset a$. It is easy to check that (—, ∼) is order reversing isomorphism and $a^{-\sim} = a^{\sim -} = a$. Since $a \wedge b = b \Subset (b \Supset a)$, then we have $(\bar{a} \wedge \bar{b})^{\sim} = (\bar{b} \Subset (\bar{b} \Supset \bar{a}))^{\sim}$, it is not difficult to prove that $(\bar{a} \wedge \bar{b})^{\sim} = (\bar{b} \Subset (\bar{b} \Supset \bar{a}))^{\sim} = a \vee b$. Hence, P is a PD-lattice. □

3. PSEUDO BOOLEAN D-POSETS

Definition 3.1. Let P be a poset with the least element 0 and the largest element 1, then P is called a pseudo boolean D-poset (PB D-poset) if there are two total binary operations on P satisfying (PB1), (PB2), (PB3) and

(PB5): For any $a, b, c \in P$, $(a \Subset b) \Supset c = (a \Supset c) \Subset b$.

Example 3.2. Let P be a l-group and u is a strong unit of P. $\Gamma(P, u) = [0, u]$ and $a \Subset b = (-b + a) \vee 0$, $a \Supset b = (a - b) \vee 0$ for any $a, b \in [0, u]$, then $(Gamma(P, u); \Subset, \Supset, 0, u)$ is a PB D-poset.

Proposition 3.3. *Let $(P; \leq, \Subset, \Supset, 0, 1)$ be a PB D-poset. If $\Subset = \Supset$, let $\ominus = \Subset = \Supset$, then $(P; \leq, \ominus, 0, 1)$ is a Boolean D-poset.*

Conversely, let $(P, \leq, \ominus, 0, 1)$ be a Boolean D-poset. Then $(P; \leq, \ominus, \ominus, 0, 1)$ is a PB D-poset.

Proposition 3.4. *A PB D-poset is a PD-lattice.*

Proof: By the Theorem 2.14, we only have to prove (PB4) holds.

First, we prove that $a \Subset a = 0$ for any $a \in P$. Since $a \Supset a = (a \Subset 0) \Supset (a \Subset 0) = (a \Supset (a \Subset 0)) \Subset 0 = (0 \Supset (0 \Subset a)) \Subset 0 \leq 0 \Supset (0 \Subset a) \leq 0$. So $a \Supset a = 0$. Analogously, $a \Subset a = 0$. Suppose $a \leq b \leq c$, then by (PB5) $(c \Subset a) \Supset (c \Subset b) = (c \Supset (c \Subset b)) \Subset a = (b \Supset (b \Subset c)) \Subset a = (b \Supset 0) \Subset a = b \Subset a$. Similarly, we can prove another case. Hence PB4 holds. □

Proposition 3.5. *Let P be a PB D-poset. Then*

(i) For any $a, b \in P$, $a \vee b \Subset a = b \Subset a \wedge b$.
$a \vee b \Supset a = b \Supset a \wedge b$.

(ii) For any $a, b \in P$, $a \Supset b = \bar{b} \Subset \bar{a}$.
$a \Subset b = \tilde{b} \Supset \tilde{a}$.

Proof:

(i) From the proof of Theorem 2.14, we know $a \wedge b = b \Supset (b \Subset a)$ and $a \vee b = (\bar{a} \wedge \bar{b})^{\sim} = (\tilde{a} \wedge \tilde{b})^{-}$. Hence, $b \Subset a \wedge b = b \Subset (b \Supset (b \Subset a)) = (b \Subset a) \Subset ((b \Subset a) \Supset b) = b \Subset a$. And $(a \vee b) \Subset a = (\tilde{a} \wedge \tilde{b})^{-} \Subset a = (1 \Supset \tilde{a} \wedge \tilde{b}) \Subset a = (1 \Subset a) \Supset (\tilde{a} \wedge \tilde{b}) = (1 \Subset a) \Supset (\tilde{b} \Subset (\tilde{b} \Supset \tilde{a})) = \tilde{a} \Supset (\tilde{b} \Subset (\tilde{b} \Supset \tilde{a})) = \tilde{a} \Supset (\tilde{a} \Subset (\tilde{a} \Supset \tilde{b})) = (\tilde{a} \Supset \tilde{b}) \Supset ((\tilde{a} \Supset \tilde{b}) \Subset \tilde{a}) = \tilde{a} \Supset \tilde{b} = (1 \Subset a) \Supset (1 \Subset b) = (1 \Supset (1 \Subset b)) \Subset a = b \Subset a$. Similarly, we can prove the second expression.

(ii) By (PB5), $\bar{b} \Subset \bar{a} = (1 \Supset b) \Subset (1 \Supset a) = (1 \Subset (1 \Supset a)) \Supset b = a \Supset b$. For another case, we can similarly prove. □

Corollary 3.6. *Let P be a PB D-poset. Then for any* $a, b \in P$, $a \vee b \Subset a = b \Subset a \wedge b$ *iff* $a \vee b \Supset a = b \Supset a \wedge b$.

Theorem 3.7. *Let* $(P; \leq, /, \backslash, 0, 1)$ *be a PD-lattice. Then* $(P; \Subset, \Supset, 0, 1)$ *is a PB D-poset iff for any* $a, b \in P$, $a\backslash(a \wedge b) = (a \vee b)\backslash b$. *Where* $a \Subset b = a\backslash(a \wedge b)$, $a \Supset b = a/(a \wedge b)$.

Proof: "If part." From Theorem 2.14 and Definition 3.1, we only have to prove (PB5).

By the definition, $(a \Subset b) \Supset c = (a \Subset b)/(a \Subset b) \wedge c = (a\backslash(a \wedge b))/((a\backslash(a \wedge b)) \wedge c)$. From the Proposition 2.4(vi), the expression is equal to $(a/((a\backslash a \wedge b) \wedge c \wedge a))\backslash(a \wedge b) = ((a/(a\backslash a \wedge b)) \vee (a/(a \wedge c))\backslash(a \wedge b) = ((a \wedge b) \vee (a \Supset c))\backslash(a \wedge b) = (a \Supset c)\backslash((a \wedge b) \wedge (a \Supset c)) = (a \Supset c) \Subset (b \wedge (a \Supset c)) = (a \Supset c) \Subset b$. i.e., PB5 is true. Hence, $(P, \Subset, \Supset, 0, 1)$ is a PB D-poset.

"Only if part." By Proposition 3.5(i), then $a\backslash(a \wedge b) = a\backslash(a \wedge a \wedge b) = a \Subset a \wedge b = (a \vee b) \Subset b = (a \vee b)\backslash((a \vee b) \wedge b) = (a \vee b)\backslash b$. □

Theorem 3.8. *Every pseudo MV-algebra is a PB D-poset, and conversely, every PB D-poset is a pseudo MV-algebra.*

Proof: Let $(M; \oplus, -, \sim, 0, 1)$ be a pseudo MV-algebra. Define $a \Supset b = (b \oplus \tilde{a})^-$, $a \Subset b = (\bar{a} \oplus b)^\sim$. By definition, $a \Supset 0 = (0 \oplus \tilde{a})^- = a$, $a \Subset 0 = (\bar{a} \oplus 0)^\sim = a$. That is, (PB1) holds. For (PB2), since $a \Subset (a \Supset b) = a \Subset (b \oplus \tilde{a})^- = (\bar{a} \oplus (b \oplus \tilde{a})^-)^\sim = (b \oplus \tilde{a}) \odot a = b \odot (\bar{b} \oplus a)$, $b \Subset (b \Supset 0) = b \Subset (a \oplus \tilde{b})^- = (\bar{b} \oplus (a \oplus \tilde{b})^-)^\sim = (a \oplus \tilde{b}) \odot b = a \odot (\bar{a} \oplus b) = b \odot (\bar{b} \oplus a) = a \Subset (a \Supset b)$. Since — and $\sim$ are order reversing, it is easy to see (PB3) is true. Now, we prove (PB5). Because $(a \Subset b) \Supset c = (\bar{a} \oplus b)^\sim \Supset c = (c \oplus (\bar{a} \oplus b)^{\sim\sim})^- = (c^{-\sim} \oplus (\bar{a} \oplus b)^{\sim\sim})^- = (\bar{a} \oplus b)^\sim \odot \bar{c} = (\tilde{b} \odot a) \odot \bar{c}$, and $(a \Supset c) \Subset b = ((c \oplus \tilde{a})^{--} \oplus b^{\sim -})^\sim = \tilde{b} \odot (c \oplus \tilde{a})^- = \tilde{b} \odot (a \odot \bar{c}) = (\tilde{b} \odot a) \odot \bar{c} = (a \Subset b) \Supset c$, hence, (PB5) holds.

Conversely, let $(P; \Subset, \Supset, 0, 1)$ be a PB D-poset. Define $a \oplus b = (\bar{b} \Supset a)^\sim$, $a \odot b = a \Supset b$ for any $a, b \in P$. Where $\bar{a} = 1 \Supset a$, $\tilde{a} = 1 \Subset a$.

First, we prove $a \oplus b = (\bar{b} \Supset a)^\sim = (\tilde{a} \Subset b)^-$; $a \odot b = a \Supset \tilde{b} = b \Subset \bar{a}$.

Indeed, let $(\bar{b} \Supset a)^\sim = u$, then $\bar{b} \Supset a = \bar{u}$. Since $\bar{b} \wedge a = \bar{b} \Subset (\bar{b} \Supset a)$, i.e., $\bar{b} \wedge a = \bar{b} \Subset \bar{u}$, by Proposition 3.5(ii), it follows that $u \Supset b = \bar{b} \wedge a$. Hence, $(u \Supset b)^\sim = (\bar{b} \wedge a)^\sim = b \vee \tilde{a}$. Thus by Proposition 3.5(i), $(1 \Subset (u \Supset b)) \Subset b = b \vee \tilde{a} \Subset b = \tilde{a} \Subset (\tilde{a} \wedge b)$. That is $(1 \Subset (u \Supset b)) \Subset (u \Subset (u \Supset b)) = \tilde{a} \Subset (\tilde{a} \Supset (\tilde{a} \Subset b)) = \tilde{a} \Subset b$, So, $\tilde{u} = \tilde{a} \Subset b$ which implies that $a \oplus b = (\bar{b} \Supset a)^\sim = u = (\tilde{a} \Subset b)^-$.

From Proposition 3.5(ii), it is not difficult to see that $a \odot b = a \Supset \tilde{b} = b \Subset \bar{a}$.

From the two expression, we can conclude:

(i) For any $a, b \in P$, $(a \odot b)^\sim = \tilde{b} \oplus \tilde{a}$; $(a \odot b)^- = \bar{b} \oplus \bar{a}$.
(ii) For any $a, b \in P$, $(a \oplus b)^\sim = \tilde{b} \odot \tilde{a}$; $(a \oplus b)^- = \bar{b} \odot \bar{a}$.

Now, we prove (A1)–(A8) in Definition 1.5 is true.

Obviously, from the definition and the properties of $\Subset$, $\Supset$, (A2) and (A3) hold. And (A4) and (A8) are true by the properties of $\Subset$, $\Supset$. For (A5), since $(\bar{x} \oplus \bar{y})^{\sim} = (x \Subset \bar{y})^{-\sim} = x \Subset \bar{y} = y \Supset \tilde{x}$, and similarly, $(\tilde{x} \oplus \tilde{y})^{-} = y \Supset \tilde{x}$. That is (A5) holds. From the proof of theorem 2.14, for any $a, b \in P$, $a \vee b = (\bar{a} \wedge \bar{b})^{\sim} = (a \Supset b) \oplus b$, and $a \vee b = (\tilde{a} \wedge \tilde{b})^{-} = b \oplus (a \Subset b)$. By definitions, we have $x \oplus (\tilde{x} \odot y) = x \oplus (y \Subset x) = x \vee y$, $y \oplus (\tilde{y} \odot x) = y \oplus (x \Subset y) = x \vee y$, $(x \odot \bar{y}) \oplus y = (x \Supset y) \oplus y = x \vee y$, $(y \odot \bar{x}) \oplus x = x \vee y$. So (A6) holds. Analogously, since for any $a, b \in P$, $a \wedge b = a \Supset (a \Subset b) = a \Subset (a \Supset b)$, and $x \odot (\bar{x} \oplus y) = x \odot (x \Subset y)^{-} = x \Supset (x \Subset y) = y \wedge x$, $(x \oplus \tilde{y}) \odot y = (y \Supset x)^{\sim} \odot y = y \Subset (y \Supset x) = y \wedge x$, hence, (A7) is true.

Now, we prove (Al). First, we prove for any a, b, $c \in P$, $(a \odot b) \odot c = a \odot (b \odot c)$.

Because of by definition, $(a \odot b) \odot c = (a \Supset \tilde{b}) \odot c = c \Subset (a \Supset \tilde{b})^{-} = c \Subset (1 \Supset (a \Supset \tilde{b})) = c \Subset (1 \Supset (b \Subset \bar{a})) = (1 \Supset (b \Subset \bar{a}))^{\sim} \Supset \tilde{c} = (1 \Subset (1 \Supset (b \Subset \bar{a}))) \Supset \tilde{c} = (b \Subset \bar{a}) \Supset \tilde{c} = (b \Supset \tilde{c}) \Subset \bar{a} = a \odot (b \odot c)$. But $a \oplus (b \oplus c) = a^{\sim -} \oplus (b \oplus c)^{\sim -} = ((b \oplus c)^{\sim} \odot \tilde{a})^{-} = ((\tilde{c} \odot \tilde{b}) \odot \tilde{a})^{-} = (\tilde{c} \odot (\tilde{b} \odot \tilde{a}))^{-} = (\tilde{b} \odot \tilde{a})^{-} \oplus c^{\sim -} = (a^{\sim -} \oplus b^{\sim -}) \oplus c^{\sim -} = (a \oplus b) \oplus c$.i.e., (Al) holds. Thus $(P; \oplus, \odot, \sim, -, 0, 1)$ is a pseudo MV-algebra. By the above, let define two binary operations $\trianglerighteq$, $\trianglelefteq$ on the pseudo MV-algebra $(P; \oplus, \odot, \sim, -, 0, 1)$ by the formula $a \geq b = (b \oplus a^{\sim})^{-}$, $a \trianglerighteq b = (a^{-} \oplus b)^{\sim}$. Then $(P; \trianglerighteq, \trianglelefteq, 0, 1)$ is a PB D-poset in which for every a, $b \in P$, there holds $a \trianglelefteq b = (b \oplus \tilde{a})^{-} = a \Supset b$, $a \trianglerighteq b = (\bar{a} \oplus b)^{\sim} = a \Subset b$, therefore, the operations $\trianglelefteq$, $\trianglerighteq$ and $\Supset$, $\Subset$ are identical respectively. □

Corollary 3.9. *PB D-posets are intervals in l-groups (Dvurečenskij, 2002).*

4. IDEALS IN PD-POSETS

Definition 4.1. Let P be a PD-poset. A nonempty subset $I \subseteq P$ is called an ideal in P iff

(I1) $a \in I, b \in P, b \leq a$, then $b \in I$.
(I3) $a \in I, b \in P, a \leq b, b \backslash a \in I$ or $b/a \in I$, then $b \in I$.

It is easy to see that (I3) is equivalent to (I2):
$a \in I, b \in I$, and $a + b$ exists, then $a + b \in I$.

Definition 4.2. Let P be a PD-lattice. A D-ideal is a subset I of P which satisfies (I3) and the following :
(R) If $a \in I$, then for any $b \in P$, $(a \vee b) \backslash b \in I$, and $(a \vee b)/b \in I$.

Theorem 4.3. *Let P be a PD-lattice. A subset I of P is a D-ideal iff it is a Riesz ideal.*

Proof: Suppose I is a Riesz ideal, we only have to prove (R) is true.

Given $a \in I$, and any $b \in P$, set $c = (a \vee b)\backslash b$. Since $b + c = a \vee b \geq a$, by (RO), we can find $h, k \in I$ with $h \leq b$, $k \leq c$ and $h + k \geq a$. Obviously, $b + k$ exists, since $b + k \geq h + k \geq a$, we have $b + k \geq a \vee b$, i.e., $b + k \geq b + c$. Thus, $k = c$. Hence, $c \in I$. Similarly, we can prove $(a \vee b)/b \in I$, i.e.,(R) is true.

Conversely, assume that I is a D-ideal, we first prove (I1). For $a \in I$, take $b \in P$, $b \leq a$. Let $c = a/b$, then $(a \vee c)\backslash c = a\backslash c = b$, hence, $b \in I$ by (R).

For (R0). Given $a \in I$, c, $d \in P$, $c + d$ exists, $a \leq c + d$, set $h = (a \vee d)/d$, $k = (a \vee h)\backslash h$ such that $h, k \in I$ by (R). Then $h \leq (a \vee (c + d))/d = (c + d)/d = c$. Moreover, since $h + d = a \vee d \geq a$, we have $k \leq (a \vee (h + d))\backslash h = d$. Finally, $h + k = a \vee h \geq a$, which complete the proof. □

Corollary 4.4. *An ideal I in PD-lattice is a D-ideal iff $b \in I$ implies $\phi_l(a, b) = a/(a \wedge \bar{b}) \in I$ and $\phi_r(a, b) = a\backslash(a \wedge \tilde{b}) \in I$. i.e., An ideal in PD-lattice is a D-ideal iff it is closed under generalized Sasaki projections. (Pulmannová, 2003).*

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